METHODS FOR JUSTIFYING AND REFINING THE THEORY OF SHELLS
(Survey of recent results)

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We consider the problem of reducing the three-dimensional equations of the theory of elasticity to two-dimensional shell equations, a problem which has excited considerable interest during the last decade.

There are numerous results in this area, and they will be discussed here only from a general point of view, without details. A more detailed treatment may be found in surveys [1 – 4].

1. Suppose that the problem of studying the state of stress of a thin elastic shell, within the framework of the linear theory of elasticity, may be posed as a three-dimensional boundary value problem

$$M(\Phi) = 0, \; M^f(\Phi) = M_0^f, \; M^b(\Phi) = M_0^b \tag{1}$$

where the equations represent, respectively, the system of equations of the theory of elasticity, the conditions on the surfaces, and the conditions on the edges of the shell. Here Φ represents the set of unknown quantities, for example the stresses and displacements. Then the task at hand consists in constructing a sequence of two-dimensional boundary value problems

$$L_j(\Psi_1, \Psi_2, \ldots, \Psi_j) = L_{j,0}$$
$$L_j^b(\Psi_1, \Psi_2, \ldots, \Psi_j) = L^b{}_{j,0} \; (j = 1, 2, \ldots) \tag{2}$$

where the first equation represents a two-dimensional system of equations, the second represents the boundary conditions, and the Ψ_j are some new set of unknown quantities.

The connection between (1) and (2) consists in the statement that certain expressions H_j, composed in a definite manner from the solutions Ψ_j of the two-dimensional problems, should in some sense approach the solution of the three-dimensional problem as $j \to \infty$

$$H_j(\Psi_1, \Psi_2, \ldots, \Psi_j) \to \Phi$$

To solve this problem many approaches have been tried which will not be described here. However, if we neglect details, the vast majority of the methods may be grouped into three categories: (a) the method of hypotheses; (b) the method of expansion with respect to thickness; and (c) asymptotic method.

2. The method of hypotheses consists in imposing certain assumptions on the character of the function Φ, these assumptions being chosen so that they allow a two-dimensional formulation as a boundary-value problem

$$L(\Psi) = L_0, \quad L^b(\Psi) = L_0^b$$

which is in some sense equivalent to the problem (1).

This method does not lead to a sequence of boundary value problems, but rather to some definite boundary value problem, which may be varied only in its second order terms, and only according to the accuracy with which the given hypothesis is realized.

The simplest and most important example of a theory constructed by the method of hypotheses is the classical theory, which was developed only at the end of the last century and which has not lost its significance since that time. It is based on the Kirchhoff-Love hypotheses, and in this case the symbol Ψ would represent, for example, the forces, moments, and displacements of the mean surface.

Other hypotheses have also been formulated, less rigid than those of Kirchhoff-Love [5 – 9]. Further on in connection with this we shall speak of a method of weakened hypotheses.

3. The advantages and shortcomings of the method of hypotheses are very significant and evident: the advantages consist in the clarity and relative simplicity of the final form; the disadvantages are found in the impossibility of increasing the accuracy (without changing the hypotheses) and in the difficulty in obtaining error estimates.

4. Recently a great amount of attention has been focused by foreign scholars on the work of Koiter [10 and 11] and John [12], who undertook anew to eliminate the second of the indicated disadvantages for the classical theory of shells (among the earlier papers dealing with this question, we recall [13]).

Koiter and John modified the Kirchhoff-Love hypotheses in that they relaced them, roughly speaking, by assumptions on the planar character of the stress function (i. e. by the assumption that all stresses not directed parallel to the tangent plane of the mean surface are relatively small). Working under this new formulation, which, though new, does not change the essential points, they obtained an estimate for the error in the Kirchhoff-Love hypotheses (not only for linear, but also for nonlinear problems), which, staying closer to Koiter's formulation, may be expressed by Formula

$$\varepsilon = \max \{O(h^1{}_*), \quad O(h_*{}^{2-2t})\} \tag{3}$$

where h_* is the relative thickness, and t is the variability index.

It should be noted that Koiter and John estimate only those errors which arise in passing from the three-dimensional equations of the theory of elsticity to the two-dimensional equations of shell theory, but do not consider the errors connected with going from the three-dimensional boundary conditions to the two-dimensional ones. Therefore their results, though very important, strictly speaking, are unsufficient to account for the errors obtained in shell calculations, i. e. in solving the corresponding two-dimensional boundary value problems. We shall shortly come back to this point.

5. The method of expanding with respect to thickness consists in representing Φ in the form

$$\Phi = \Sigma \varphi_n(\gamma)\, \Phi_n(\alpha, \beta)$$

where γ is the transverse coordinate and α, β are coordinates in the mean surface.

The unknown quantities Ψ_j in this method are taken to be the coefficients Φ_n for which a sequence of boundary value problems (2) are formed by some method or another.

The functions $\varphi_n(\gamma)$ are most often chosen in the form of powers of γ (see [14 – 20]) or Legendre polynomials [21 – 24].

6. By asymptotic methods we mean those in which the smallness of the thickness of the shell is used most fully. In this approach one deals at each stage of the calculations only with quantities of the same order of magnitude relative to powers of h_*. Thus, basically, in all papers based on an asymptotic method [25 – 37],

$$\Phi = h_*^{\,a} \sum_{s=0}^{S} h_*^{\,s} \Phi_s \tag{4}$$

The essential difference between the asymptotic method and the method of expansion with respect to thickness appears in the structure of the boundary value problems (2).

In the method of expansion one obtains boundary value problems (2) of general type; thus for fixed j the entire group of unknowns $\Psi_1, \Psi_2, \ldots, \Psi_j$ must be determined simultaneously, by solving a boundary value problem whose order increases with increasing j.

In the asymptotic method the sequence of boundary value problems (2) assumes the form

$$L(\Psi_j) = F_j(\Psi_1, \Psi_2, \ldots, \Psi_{j-1}), \quad L^b(\Psi_j) = F_j^{\,b}(\Psi_1, \Psi_2, \ldots, \Psi_{j-1})$$

where F_j and $F_j^{\,b}$ are known expressions involving the functions indicated.

This means that the boundary value problems (2) in the asymptotic method have an iterative character, i.e. the solution process consists in a j-fold solution of boundary value problems, the problems differing among themselves only in the meaning of the known functions entering into the right sides of the equations and boundary conditions.

7. The method of hypotheses, the method of expansion with respect to thickness, and the asymptotic method are directed toward the solution of one and the same problem; and regardless of their apparent differences, should lead to similar results. The various schemes for constructing a general theory of shells will be considered in more detail from this point of view.

8. The simplest, and at the same time the most important property of the state of stress of a thin shell is the fact that it can be separated into an internal state of stress distributed, as a rule, throughout the shell, and a boundary layer state of stress localized near the edge of the shell.

9. The differential equations of the classical theory have no integrals corresponding to a boundary layer. Thus it follows that the assumptions imposed at the basis of the classical theory exclude the entire boundary layer, retaining only the internal state of stress. This property may be taken as decisive for Kirchhoff-Love-type hypotheses (in this sense we mentioned earlier that the assumptions of Koiter and John do not differ essentially from the hypotheses of Kirchhoff-Love type).

10. The method of expansion with respect to thickness and the method of weakened hypotheses lead to equations of higher order than those of the classical theory. The increase in order is obtained through extra terms with small (as $h_* \to 0$) coefficients. An asymptotic analysis of the partial differential equations with this type of perturbation [38 – 40] shows that if the perturbing terms lead to an increase in the order of the equation, then their effects are first, that they produce small changes in those integrals belonging to the unperturbed equations, and secondly, they generate new integrals with sharp

variation. The latter correspond to a boundary layer (thus by weakened hypotheses one understands assumptions which retain the boundary layer in some sense).

Note. The classical shell equations also have decaying solutions, corresponding to the so-called edge effect, known already to Love. The ordinary layer differs from them in that it has a large variability index t (for the edge effect $t \leqslant 1/2$ and for the boundary layer $t = 1$).

11. The considered property of the state of stress of a shell appears most clearly in the asymptotic method.

Various representations for the solution in the form (4), due to different choices for the exponent a_k, correspond to different iterative processes for solving the equations. They deal with basic iterative processes, allowing the construction of internal state of stress, and also auxiliary iterative processes for the solution in the boundary layer.

It is shown in [28 and 30] that to satisfy the boundary conditions formulated in terms of the three-dimensional theory, one must use the available freedom not only of the basic, but also of the auxiliary iterative process. From this it follows, as a rule, that the state of stress of the shell will be separated into an internal state of stress and a boundary layer.

12. Thus, the method of weakened hypotheses, the method of expansion with respect to thickness, and the asymptotic method agree qualitatively one with another as regards the particular property of the state of stress mentioned above. This general assertion has been verified by a concrete asymptotic analysis of the differential equations for the deflection of a plate. This was carried out in [24] for the equations constructed by the method of expansion with respect to thickness, and in [41] for the equations constructed by the method of weakened hypotheses. However, it should not be thought that the difference between the asymptotic method on the one hand, and the method of expansion , or the method of relaxed hypotheses on the other, reduces only to the question of the stage at which an asymptotic analysis of the equations is performed. On the edges between the narrow sides of the shell and its faces, there will appear singuliarities in the solution of the three-dimensional problem. In the method of expansion or the method of relaxed hypotheses they are "smeared out", i.e. approximated by continuous functions; whereas in the asymptotic method they are "singled out" and become singuliarities in solutions of the boundary layer equations.

13. Let us examine the physical content of the concepts of the internal state of stress and the bounday layer.

The nature of the internal state of stress is clear: it is that state of stress which is obtained in first approximation when the shell is treated according to the classical theory. It is produced by external surface forces and a portion of the boundary forces which are nonself-balancing(here and below we include reactive forces from supports among the boundary forces).

The boundary layer is produced by the portion of boundary forces which are self-balancing through the thickness of the shell. On the edge of the shell we fix some normal n to the mean surface and consider the section of the body in the form of a curved strip η produced by slicing it by a plane through n orthogonal to the edge (Fig. 1). Then, roughly speaking, the boundary layer near n may be treated as the state of stress arising in the strip η from the action on its end piece of some self-balancing force system. According to the principle of Saint-Venant, the boundary layer decays rapidly away from the edge, and the curved strip may in first approximation be replaced by a straight strip. Such an interpretation appears superficial, but is verified by the fact that the boundary layer constructed by the asymptotic method in the first approximation reduces to the search for decaying solutions of plane and antiplane problems of the theory of elasticity in a straight semistrip [23 – 37] (see also Appendix).

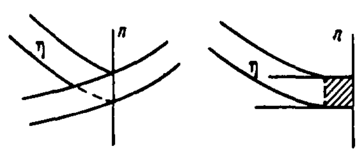

Fig. 1

14. The internal state of stress and the boundary layer may be distinguished from one another not only by their physical significance, but also by the mathematical formulation of the corresponding problems. Therefore any refined shell theory should be judged according to its capacity for describing the boundary layer as well as the internal state of stress. Such a test would appear difficult for most any theory. In particular, from this point of view neither the method of expansion with respect to thickness nor the method of relaxed hypotheses are totally effective. They describe well the internal state of stress in which the stresses and displacements vary through the thickness according to a simple law, but are ill suited to describing the boundary layer, in which the displacement profile near the edge may have a very strange appearance.

15. In connection with the foregoing, the idea of a separate study of the theory of internal state of stress and of the theory of the boundary layer merits consideration.

A theory of the internal state of stress would arise as a direct generalization and refinement of the classical theory. We shall speak below of this in more detail. The theory of the boundary layer, or, if desired, the theory of edge effects, should be constructed as new. This problem apparently must be treated as the problem of constructing decaying solutions of plane and antiplane problems of the theory of elasticity for semistrips.

16. A clear way to use the idea of separation of the basic state of stress and boundary layer is not available in every problem in the theory of shells; however in linear problems of the static equilibrium of shells, approaches based on this idea are very obvious. It is shown in [23 – 37] that if one is satisfied with a first approximation for the basic state of stress and for the boundary layer, then the treatment of the shell takes place in two stages: the first consists in proceeding according to the classical theory, and the second, in constructing the boundary layer. The first stage here does not depend at all on the second, and this means that all calculations carried out within the framework of the classical theory retain their validity, but are now seen to be calculations which in certain cases must be supplemented by the construction of a boundary layer.

17. The interaction between the internal state of stress and the boundary layer appears only in the process of finding boundary conditions on the lateral edges. To say it another way, it reduces to a "distribution of responsibilities": The internal state of stress reacts with those active and reactive boundary forces which are nonself-balancing through the thickness, whereas the boundary layer is concerned only with the self-balancing part of these forces.

Thus in forming the complete state of stress of the shell, the boundary layer plays a dual role: firstly, it produces a local change in the total state of stress, added near the edge to the internal state of stress; secondly, it infuences the internal state itself by changing the boundary conditions which the latter should satisfy. The question naturally arises as to how significant each of these two effects is.

18. It has been shown that at the edge the maximal stresses in the boundary layer are commensurable in intensity with those of the internal state [30, 34, 36 and 41]. Therefore the problems in which it is important to study the state of stress of the edge zone, for example the problems of concentration of stresses, belong to the class of problems for which the calculations must be completed by constructing a boundary layer.

19. The influence of the boundary layer on the internal state of stress is no longer strictly an edge effect; it leads to a total change. One of the methods of estimating the magnitude of this effect consists in trying to completely separate out the consideration of the problem of the internal state of stress, so that the influence of the boundary layer is expressible in the form of a correction to the boundary conditions. This is hardly possible in all accuracy, but may be done within the framework of a certain number of assumed approximations. Moreover, Friedrichs showed [25] that the effect of the boundary layer is already considered in the classical theory. Namely, the known modification of the boundary conditions in plate theory due to the introduction of the cross-acting forces mentioned is the very effect of including a boundary layer. In [42] it is shown that the mentioned normal and shear forces have the same sort of meaning as regards the boundary conditions in shell theory.

A somewhat more accurate explanation of the formulation of boundary conditions in classical shell theory and its relation to the boundary layer is given in Appendix.

20. The influence of the boundary layer on boundary conditions for the internal state of stress may be described even more exactly than was done by Friedrichs, Green and Laws.

For the theory of deflection of plates this was done in [43], where the following boundary conditions for the internal state of stress were derived:

free edge $(\sigma = \tau_t = \tau_n = 0)$

$$M_\alpha + \left[Ah \frac{\partial H}{\partial s_\beta}\right] = 0, \qquad N_\alpha + \frac{\partial H}{\partial s_\beta} + \left[Ah \frac{\partial}{\partial s_\beta}(kH)\right] = 0$$

hinged edge $(\sigma = \tau_t = w = 0)$

$$M_\alpha + \left[Ah \frac{\partial H}{\partial s_\beta} + BkhM_\beta\right] = 0, \qquad w = 0 \tag{5}$$

hinged edge $(\sigma = v = w = 0)$

$$M_\alpha = 0, \quad w = 0$$

rigidly clamped edge $(u = v = w = 0)$

$$D \frac{\partial w}{\partial s_\alpha} - \left[\frac{C\nu}{1-\nu^2} \frac{h}{2}(M_\alpha + M_\beta)\right] = 0$$

The terms on the left of these equations, exclusive of the bracketed terms, represent the well known boundary conditions from the classical theory of the deflection of plates. The bracketed terms yield the influence of the boundary layer; in these terms k is the curvature of the edge, ν is Poisson's ratio, and A, B, C are numbers whose determination can only be made by solving certain plane and antiplane problems from the theory of elasticity $A \approx 1.26, \qquad B \approx -0.0083, \qquad C \approx -0.0917 \quad \text{for} \quad \nu = {}^1/_3$

The parentheses following the designation of the manner the edge is fastened indicate the corresponding boundary conditions in the three-dimensional theory (σ is normal stress and τ_t, τ_n are shear stresses directed along the tangent and normal to the middle surface, respectively).

21. From (5) it is clear that in plate theory the influence of the boundary layer is expressed by terms of the order h_*, where h_* is the ratio of semithickness to a typical dimension in the shape of the plate.

If the variability of the state of stress along the edge is large, then the influence of the boundary layer due to terms containing derivatives is increased to result in quantities of the order

$$\varepsilon = O(h_*^{1-t}) \tag{6}$$

In Koiter's and John's works the assumption of the planar character of the state of stress is essentially a hypothesis of Kirchhoff-Love type: it eliminates the boundary layer. This means that their estimate of the error in the shell equations (3) may be considered as an error estimate for boundary problems in shell theory under the additional assumption that the influence of the boundary layer does not exceed the order in (3). Such an assumption is not always satisfied, as is evident from a comparison between (3) and (6).

22. In formulating shell problems as boundary value problems, one often uses a very rough schematization of the physical conditions at the edge of the shell. The refinement of the boundary conditions metioned above (5) allows us to judge the possible consequences of so doing. For comparison purposes two variants of the three-dimensional boundary conditions are given, corresponding to so-called hinged support. They lead to identical boundary conditions within the framework of the classical theory of the deflection of plates, but the correction terms are totally different. This means that in refining the theory of shells one must maintain sufficient accuracy even in formulating the edge conditions.

23. Returning to a comparison of the asymptotic method and the method of expansion with respect to thickness, we note that the second of these methods has greater generality than the first.

The method of expansion with respect to thickness is applicable even in the case when the thickness of the shell is not small. Moreover, in this method no assumptions are made beforehand regarding the nature of the state of stress sought. The asymptotic method is based essentially on the smallness of the thickness of the shell and it involves certain implicit assumptions about the nature of the state of stress. They are made when the choice is made for the number a in expansion (4). The advantages of generality are obvious. They are on the side of methods of expansion with respect to thickness, which are probably also preferable, for example, in such problems as the construction of a theory of shells for moderate thickness. However, this generality is attained at the price of increasing the complexity, and it often proves advantageous to restrict one's attention to sufficiently simple cases. In this sense the advantage lies on the side of the asymptotic method.

An example of the use of this advantage was already given above: an appropriate choice of a in expansion (4) makes it possible to separate the internal state of stress from the boundary layer.

24. Each rigorously constructed two-dimensional theory of thin shells is equivalent to a formulation of some properties of the solution of three-dimensional boundary value problems of the theory of elasticity, applied to narrow regions. It is hardly to be expected that these properties will be of a sufficiently simple form if certain limitations are not introduced when posing the question. Therefore one may expect that a sufficiently simple two-dimensional theory will be a specialized theory, i.e. a theory which is intended for only a specific class of problems. As will be shown below, the asymptotic method yields the better possibilities for this.

25. We shall consider in greater detail the theory of the internal state of stress, proceeding with the asymptotic method and with the view in mind of examining more closely the connections between this theory and the classical theory.

One of the possible variants in the choice of the exponent in (4) is such that in expanded form the equation will read

$$(\sigma_\alpha,\ \tau_{\alpha\beta},\ \sigma_\beta) = h_*^{-1} \sum h_*^{s/q} (\sigma_\alpha^{(s)},\ \tau_{\alpha\beta}^{(s)},\ \sigma_\beta^{(s)})$$

$$(\tau_{\alpha\gamma},\ \tau_{\beta\gamma}) = h_*^{-t} \sum h_*^{s/q} (\tau_{\alpha\gamma}^{(s)},\ \tau_{\beta\gamma}^{(s)}), \qquad \sigma_\gamma = h_*^{c} \sum h_*^{s/q} \sigma_\gamma^{(s)} \tag{7}$$

$$(u_\alpha,\ u_\beta) = h_*^{-1+t} \sum h_*^{s/q} (u_\alpha^{(s)},\ u_\beta^{(s)}), \qquad u_\gamma = h_*^{-1-c} \sum h_*^{s/q} u_\gamma^{(s)}$$

where $c = 0$ for $t \leqslant 1/2$ and $c = 1 - 2t$ for $1/2 \leqslant t < 1$, and the number q is determined from the equation $t = p/q$, in which p and q are integers.

By this choice a certain limited class of problems is considered, since Formula (7) predetermines the asymptotics of the stress. In particular, expansion (7) does not correspond to a boundary layer, and the theory based on (7) shouls be considered as a theory of the internal state of stress. Moreover, it must be considered a specialized theory of internal ststes since the class of internal states of stress compatible with (7), although very wide (for more details on this see [44]), does not embrace all possible cases.

With the aid of (7), two-dimensional shell equations are constructed as follows. One fixes the number of terms to be retained in the expansion (7). The formulas obtained are put into the three-dimensional equations of the theory of elasticity and the usual procedure of equating coefficients of like powers of h_* is carried out. This leads to a system of approximate equations. In these equations one easily integrates with resect to the transverse coordinate and, having thus eliminated it, is left with two-dimensional equations. Furthermore, using such concepts as forces, moments, displacements of the mean surface, etc., one may establish that the equations obtained are equivalent to one or another of the systems of equations of the theory of shells. If one retains the small terms in (7), then one obtains equations of approximate theories (such as a membrane theory, a theory of approximate states with large variation, and so on). An inrease in the number of terms in Expansion (7) will correspond to a transition to one variant or another of the equations of the full theory of shells. These variants will become more complex as the number of terms retained in (7) is increased, and at some instant there will occur a qualitative change in the theory (an increase in the order of the equations, a necessity of introducing new concepts, etc.).

Thus for each fixed number of terms retained in (7), there corresponds some two-dimensional shell theory possessing two important properties.

a) An estimate of its formal asymptotic accuracy is obtained immediately by seeing the order of the terms excluded from (7).

b) Such a theory of shells may be considered as initial approximation for some iterative process which allows one to obtain any formal asymptotic accuracy desired.

26. Among all variants of the classical theory obtained by the method described in the preceding Section, it is natural to select the asymptotically optimal variant, by which one undestands a variant which, on the one hand, does not leave the framework of the usual concepts of the classical theory, and on the other hand, involves the retention in (7) of the largest number of terms.

Such a variant of the classical theory was constructed in [44].

The corresponding system of equations consists of the usual number of static and purely geometrical equations and of elasticity relations having the form

$$T_1 = \frac{2Eh}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2) - \frac{\nu h}{1-\nu} m, \qquad S_1 = \frac{Eh}{1+\nu}\omega$$

$$G_1 = -\frac{2Eh^3}{3(1-\nu^2)}\left[\varkappa_1 + \nu\varkappa_2 - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\varepsilon_1 - \frac{\nu}{1-\nu}\left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)(\varepsilon_1 + \varepsilon_2)\right] - \frac{h^2\nu}{3(1-\nu)} Q_\gamma$$

$$H_1 = \frac{2Eh^3}{3(1+\nu)}\left(\tau - \frac{\omega}{2R_1}\right) \tag{8}$$

(with analogous relations for quantities with index 2).

Here Q_γ and m are the magnitude of the normal load and the normal compression, respectively.

The normal asymptotic error in the theory (8) is determined by Formula

$$\varepsilon = O(h_*^{2-2t}) \tag{9}$$

which, of course, refers only to the errors in the equations. To estimate the error in the boundary conditions one must have available formulas of the type (5), which up to now have been constructed only for plates.

27. The estimation formulas (9) and (3) are formally in agreement, but the first of them contains more information.

We consider, for example, a shell with positive curvature, loaded and supported in such a manner that its fundamental state of stress has zero variability. For such a shell the internal state of stress consists of a basic state of stress, for which by assumption $t = 0$ and a simple edge effect, for which always $t = {}^1/_2$. Therefore Formula (3) in this case yields

$$\varepsilon = O(h^1{}_*)$$

and at the same time Formula (9) gives for the basic state of stress

$$\varepsilon = O(h_*^{2-2t})$$

and for the simple edge effect

$$\varepsilon = O(h^1{}_*)$$

Formally the results coincide, since the error in the large is equal to the largest of the possible errors. However, for an evaluation of the construction the basic state of stress is more important and one should not neglect the possibility of attaining a greater accuracy for it. It should be remembered, moreover, that in this case it is a question of a theory for the internal state of stress, from which edge stresses corresponding to a boundary layer have already been excluded. It would be logically inconsistent to place other edge stresses corresponding to the simple edge effect in a position comapable to that of the basic state of stress (see Note in Section 10).

Appendix. In conclusion we consider in greater detail the connection between the boundary layer and the formulation of boundary conditions in the theory of the internal state of stress. To simplify the exposition we make certain assumptions. They can all be corroborated by pursuing asymptotic methods.

Suppose a three-dimensional medium is given in the form of a shell, and let (α, β, γ) be an orthogonal coordinate system in which α and β are parameters on the lines of curvature on the mean surface, A^2 and B^2 are the coefficients of the first quadratic form, and R_1 and R_2 are the principal radii of curvature. We introduce unsymmetric components of a stress tensor by the following equations (primes designate the usual stress components):

$$\sigma_\alpha=\left(1+\frac{\gamma}{R_2}\right)\sigma_\alpha', \quad \tau_{\alpha\beta}=\left(1+\frac{\gamma}{R_2}\right)\tau_{\alpha\beta}', \quad \tau_{\alpha\gamma}=\left(1+\frac{\gamma}{R_2}\right)\tau_{\alpha\gamma}'$$

$$\tau_{\beta\alpha}=\left(1+\frac{\gamma}{R_1}\right)\tau_{\beta\alpha}', \quad \sigma_\beta=\left(1+\frac{\gamma}{R_1}\right)\sigma_\beta', \quad \tau_{\beta\gamma}=\left(1+\frac{\gamma}{R_1}\right)\tau_{\beta\gamma}' \tag{10}$$

$$\sigma_\gamma=\left(1+\frac{\gamma}{R_1}\right)\left(1+\frac{\gamma}{R_2}\right)\sigma_\gamma'$$

We shall suppose that the coordinate surface $\alpha=\alpha_0$ is a free edge of the shell. On it the boundary conditions may be written as:

$$\sigma_\alpha+S_\alpha=0, \ \tau_{\alpha\beta}+T_{\alpha\beta}=0, \ \tau_{\alpha\gamma}+T_{\alpha\gamma}=0 \tag{11}$$

Here σ_α, $\tau_{\alpha\beta}$, $\tau_{\alpha\gamma}$ are the unsymmetric stress components of the internal state of stress, and S_α, $T_{\alpha\beta}$, $T_{\alpha\gamma}$ are the unsymmetric stress components in the boundary layer, which components are also defined by formulas of the form (10).

The problem spoken of in Section 17 consists in transferring to the boundary layer a portion of the edge stresses without disturbing its property of decaying.

We shall write out the equations of the theory of elasticity for the stresses and displacements in the boundary layer. They may be put into the form

$$\frac{\partial S_\alpha}{\partial\rho}+\frac{\partial T_{\alpha\gamma}}{\partial\gamma}+X=0, \qquad \frac{\partial T_{\alpha\beta}}{\partial\rho}+\frac{\partial T_{\beta\gamma}}{\partial\gamma}+Y=0$$

$$\frac{\partial T_{\alpha\gamma}}{\partial\rho}+\frac{\partial S_\gamma}{\partial\gamma}+Z=0$$

$$E\frac{\partial V_\alpha}{\partial\rho}=S_\alpha-\nu(S_\beta+S_\gamma)+s_\alpha, \qquad 0=S_\beta-\nu(S_\alpha+S_\gamma)+s_\beta \tag{12}$$

$$E\frac{\partial U_\gamma}{\partial\gamma}=S_\gamma-\nu(S_\alpha+S_\beta)$$

$$E\frac{\partial V_\beta}{\partial\rho}=2(1+\nu)T_{\alpha\beta}+t_{\alpha\beta}, \qquad E\left(\frac{\partial V_\alpha}{\partial\gamma}+\frac{\partial V_\gamma}{\partial\rho}\right)=2(1+\nu)T_{\alpha\gamma}+t_{\alpha\gamma}$$

$$E\frac{\partial V_\beta}{\partial\gamma}=2(1+\nu)T_{\beta\gamma}+t_{\beta\gamma}$$

$$\left(\frac{\partial}{\partial\rho}=\frac{1}{A}\frac{\partial}{\partial\alpha}\right)$$

The following notations are used here: S_α, $T_{\alpha\beta}$, S_β, $T_{\alpha\gamma}$, $T_{\beta\gamma}$, S_γ are the unsymmetric stress components, V_α, V_β, V_γ are the displacements, and by X, Y and Z we understand the following expressions:

$$X = k_\beta (S_\alpha - S_\beta) + k_\alpha (T_{\alpha\beta} + T_{\beta\alpha}) + \frac{\partial T\beta\alpha}{\partial\mu} + \gamma/R_1 \frac{\partial T_{\alpha\gamma}}{\partial\gamma} + \frac{2}{R_1} T_{\alpha\gamma}$$

$$Y = k_\alpha (S_\beta - S_\alpha) + k_\beta (T_{\alpha\beta} + T_{\beta\alpha}) + \frac{\partial S_\beta}{\partial\mu} + \gamma/R_2 \frac{T_{\beta\gamma}}{\partial\beta_\mu} + \frac{2}{R_2} T_{\beta\gamma} \tag{13}$$

$$Z = k_\beta T_{\alpha\gamma} + k_\alpha T_{\beta\gamma} - \frac{S_\alpha}{R_1} - \frac{S_\beta}{R_2} + \frac{\partial T_{\beta\gamma}}{\partial\mu}.$$

$$\frac{\partial}{\partial\mu} = \frac{1}{B}\frac{\partial}{\partial\beta}, \quad k_\alpha = \frac{1}{AB}\frac{\partial A}{\partial\beta}, \quad k_\beta = \frac{1}{AB}\frac{\partial B}{\partial\alpha}$$

s_α, s_β, $t_{\alpha\beta}$, $t_{\alpha\gamma}$, $t_{\beta\gamma}$ are certain expressions which will not be needed below.

The boundary layer should decay rapidly. Therefore the quantities in (12) sought should change rapidly with respect to α and γ; and this means that the principal terms in (12) will be those which are written out explicitly.

This may be easily verified by examining Expressions (13). These involve either expressions not differentiated with respect to α or γ, or terms containing small multipliers γ/R_1, γ/R_2; one may also convince himself that the quantities s_α, s_β, $t_{\alpha\beta}$, $t_{\alpha\gamma}$, $t_{\beta\gamma}$ are of second order.

Discarding all terms in (12) not written out explicitly, we obtain equations coinciding in form with the equations of the theory of elasticity for a body referred to Cartesian coordinates ρ, γ, μ, whose stresses and displacements do not depend on the coordinate μ. The problem of constructing such a stress-strain state is separated into a plane problem involving the determination of quantities

$$S_\alpha, S_\beta, S_\gamma, T_{\alpha\gamma}, V_{\alpha\gamma}, V \tag{14}$$

satisfying the first, third, fourth, fifth, sixth and eighth of Eqs. (12), and an antiplane problem consisting of the determination of the quantities

$$T_{\alpha\beta}, T_{\beta\gamma}, V_\beta \tag{15}$$

satisfying the second, seventh and ninth of Eqs, (12).

The physical interpretation of this result is described in Section 13. Eqs, (12), after second order terms have been discarded, revert to the equations describing the stress-strain state of a straightened strip η (see Fig. 1). The facial sides $\gamma = \pm h$ of this strip are free from loads, since the surface forces are considerd in the theory of the internal state of stress. Stresses are also absent at infinity (since by assumption the stresses should die out). With this in mind, one may compose the equilibrium conditions in the entrire semistrip $\rho_0 \leqslant \rho < -\infty$, $-h \leqslant \gamma \leqslant +h$ (ρ_0 corresponds to $\alpha = \alpha_0$). Considering X, Y, Z as components of a body force, we obtain the six following equations

$$\int_{-h}^{+h} S_\alpha \big|_{\rho=\rho_0} d\gamma + \int_{-h}^{+h} d\gamma \int_{-\infty}^{\rho_0} X d\rho = 0, \qquad \int_{-h}^{+h} T_{\alpha\beta} \big|_{\rho=\rho_0} d\gamma + \int_{-h}^{+h} d\gamma \int_{-\infty}^{\rho_0} Y d\rho = 0$$

$$\int_{-h}^{+h} T_{\alpha\gamma} \big|_{\rho=\rho_0} d\gamma + \int_{-h}^{+h} d\gamma \int_{-\infty}^{\rho_0} Z d\rho = 0, \qquad \int_{-h}^{+h} S_\alpha \big|_{\rho=\rho_0} \gamma d\gamma + \int_{-h}^{+h} d\gamma \int_{-\infty}^{\rho_0} [X\gamma - Z\rho] d\rho = 0 \tag{16}$$

$$\int_{-h}^{+h} T_{\alpha\beta} \big|_{\rho=\rho_0} \gamma d\gamma - \int_{-h}^{+h} d\gamma \int_{-\infty}^{\rho_0} T_{\beta\gamma}\, d\rho + \int_{-h}^{+h} \gamma d\gamma \int_{-\infty}^{\rho_0} Y\, d\rho = 0$$

$$\int_{-h}^{+h} d\gamma \int_{-\infty}^{\rho_0} T_{\alpha\beta} d\rho - \int_{-h}^{+h} d\gamma \int_{-\infty}^{\rho_0} Y\rho d\rho = 0$$

which represent necessary and (as may be expected) sufficient conditions for the existence of a decaying boundary layer.

In (16) the contour values of S_α, $T_{\alpha\beta}$, $T_{\alpha\gamma}$ may be replaced by the contour values of σ_α, $\tau_{\alpha\beta}$, $\tau_{\alpha\gamma}$. Then relations are obtained, from which a natural formulation of the boundary conditions of the theory of the internal state of stress should be derived. These will be conditions under which the internal state of stress will deserve its name; i.e. it really reflects elastic phenomena taking place far from the edge.

Eqs. (16) depend on quantities connected with the boundary layer, and, in order to interpret them as boundary conditions for the theory of the internal state of stress, one must make some sort of assumptions regarding the solution of the boundary layer problem. The simplest of them consists in assuming that it is possible to set X, Y, Z equal to zero everywhere in (16).

Then, after using (11) to replace the boundary values of S_α, $T_{\alpha\beta}$, $T_{\alpha\gamma}$ in the first four equations of (16) by boundary values of σ_α, $\tau_{\alpha\beta}$, $\tau_{\alpha\gamma}$ we obtain

$$\int_{-h}^{+h} \sigma_\alpha \Big|_{\rho=\rho_0} d\gamma = \int_{-h}^{+h} \tau_{\alpha\beta} \Big|_{\rho=\rho_0} d\gamma = \int_{-h}^{+h} \tau_{\alpha\gamma} \Big|_{\rho=\rho_0} d\gamma = \int_{-h}^{+h} \sigma_\alpha \Big|_{\rho=\rho_0} \gamma d\gamma = 0 \qquad (17)$$

These four equations form then the simplest variant of the boundary conditions for the internal state of stress theory (for a free edge). They signify the requirement that all three forces and the bending moment vanish at the edge. It is evident from the fifth of Eqs. (16) that the twisting moment need not vanish, since within the framework of our assumptions it is absorbed by the internal stresses of the boundary layer.

The simplest refinement of the boundary equations (17) may be attained by replacing the assumption taken above by two others:

a) in the fifth and sixth equations of (16) one may discard terms with X and Y;

b) for purposes of the first four of Eqs. (16), one may retain in Formulas (13) for X, Y, Z only the quantities (15) referring to the antiplane problem, and throw out all quantities (14) referring to the plane problem.

By virtue of assumption (b) we obtain from (13)

$$Z = \frac{\partial T_{\beta\gamma}}{\partial \mu} + k_x T_{\beta\gamma} = \frac{1}{AB} \frac{\partial}{\partial \beta} (A T_{\beta\gamma})$$

We insert this result into the third equation of (16), replacing $T_{\alpha\gamma}$ by $\tau_{\alpha\gamma}$ and see now what this condition implies.

We have

$$-\int_{-h}^{+h} \tau_{\alpha\gamma} d\gamma + \int_{-h}^{+h} d\gamma \int_{-\infty}^{\rho_0} \frac{1}{AB} \frac{\partial}{\partial \beta} (A T_{\beta\gamma})\, d\rho = 0$$

Noting that $d\rho = A d\alpha$ and that $T_{\beta\gamma}$ decays rapidly with ρ, and on this basis replacing B by its contour value B_0, we obtain

$$\int_{-h}^{+h} d\gamma \int_{-\infty}^{\rho_0} \frac{1}{AB} \frac{\partial}{\partial \beta} (A T_{\beta\gamma})\, d\rho = \frac{1}{B_0} \frac{\partial}{\partial \beta} \left\{ \int_{-h}^{+h} d\gamma \int_{-\infty}^{\rho_0} T_{\beta\gamma} d\rho \right\}$$

By virtue of the fifth of Eqs. (16) and assumption (a), one may express the integral in braces, first in terms of the contour value of $T_{\alpha\beta}$, and then in terms of the contour value of $\tau_{\alpha\beta}$. From this one obtains the following well-known condition:

$$N_1+\frac{1}{B}\frac{\partial H_1}{\partial\beta}=0$$

as a result of the simplest refinement of one of the four boundary conditions (17). The fifth of Eqs. (16) does not appear here as an independent condition, but rather is used only to satisfy the results obtained.

The boundary conditions (5) given above for the theory of plates are a result of further refinement of the considerations described here.

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